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First integrals for a general linear system of two second-order ODEs via a partial Lagrangian

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Abstract
The partial Noether operators and first integrals of a general system of two linear second-order ordinary differential equations (ODEs) with variable coefficients are studied by means of a partial Lagrangian. The canonical form for the general system of two second-order ordinary differential equations is invoked and all cases of this system are discussed with respect to partial Noether operators. We also tabulate the results for the special case $b(x) = c(x)$ of the system which was considered elsewhere using a Lagrangian and a partial Lagrangian. The first integrals are obtained explicitly by exploiting a Noether-like theorem with the help of partial Noether operators. This study gives a new way to construct first integrals for systems without a variational principle as not all linear equations have a Lagrangian. Physical applications to conservative and oscillator mechanical systems are given.

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1. Introduction

The relationship between Noether symmetries and first integrals has been a subject of rigorous investigation for Euler–Lagrange equations (see the works of Noether [1] and later works [2–4]). The classical Noether's theorem [1] possesses the beauty in the elegant explicit formula for the construction of the first integrals once the Noether symmetries are known. It establishes a relationship between equivalence class of symmetries and first integrals. In order to use this powerful theorem one needs a Lagrangian to obtain the Noether symmetries and to construct first integrals. There are equations that arise in applications which do not admit Lagrangians, e.g.,

$$y'' = y^2 + z^2, \qquad z'' = 0$$

and

$$y'' = y^2 + z^2, \qquad z'' = y.$$

For information on the classification of Lagrangians, the interested reader is referred to the paper [5] in which Douglas has provided the complete solution to the inverse problem for a system of two second-order ODEs (three-dimensional space). The procedure relies on the Riquier theory of systems of partial differential equations. The classification is made of all curve families that are extremal and nonextremal and the complete solution is obtained with respect to all the possible cases of the differential system. Note that in general the underlying system of two second-order equations does not have a Lagrangian. So the question is how to find first integrals in the absence of a Lagrangian for such systems. There are other methods as well to construct first integrals [2, 3, 6–12, 15]. First integrals are very important in the reduction of equations as well as due to their applications (see [14]). The theory of Noether symmetries and first integrals have been subjects of very active fields of research in the last several years. For an account of this theory we have referred to some papers cited above.

The relationship between symmetries and conservation laws without regard to a Lagrangian was given by the authors in [11]. Recently in [13, 15], the authors invoke a Noether-like theorem to construct first integrals without the use of a Lagrangian. They invoke partial Noether operators corresponding to a partial Lagrangian of a partial Euler–Lagrange system. In this approach, there is an explicit formula similar to the Noether formula for the construction of the first integrals once the partial Noether operators are known.

Systems of two second-order ODEs arise in relativity, classical mechanics, nonlinear oscillations, quantum and fluid mechanics, etc. Some important algebraic works have been done relating to a system of two second-order ODEs. The Lie point symmetries of a system of two second-order ODEs with constant coefficients were obtained by Gorringe and Leach [16]. The variable coefficient case of this system was later studied by Wafo and Mahomed [17]. The point symmetry properties of a Lagrangian system with two degrees of freedom were considered by Sen [18]. The symmetries of the Hamiltonian system with two degrees of freedom were also investigated by Damianou and Sophocleous [19]. In [20], Damianou and Sophocleous have obtained the Noether point symmetries for a three degrees of freedom Lagrangian system and the results for one and two degrees of freedom were also reviewed in their paper. A similar approach is adapted in this paper for the classification of partial Noether operators. The difference is that they have considered a nonlinear system whereas we are dealing with a general linear system of two second-order ODEs with variable coefficients. In [21], Naeem and Mahomed showed that the first integrals corresponding to the Noether and the partial Noether operators for a particular linear system of two second-order ODEs with variable coefficients are the same. The difference arises in the guage terms only. This result is a special case of the general case considered here. We have thus included it in table 1. The classification of partial Noether operators and first integrals for a conservative system with two degrees of freedom was also attempted in Naeem and Mahomed [22]. The algebraic criteria for linearization via point transformations for a system of two second-order ODEs was considered in [23]. The canonical forms for a system of two second-order ODEs were deduced by Wafo and Mahomed [24]. Moreover Fels in [26], considered the equivalence problem for a system of two second-order ODEs. The linearizability criteria for a system of two second-order quadratically semi-linear ODEs using invertible transformations were investigated by Mahomed and Qadir [25].

The objective of this paper is to construct the partial Noether operators and first integrals of a general linear system of two variable coefficient equations that in general do not admit a standard Lagrangian. Since partial Lagrangians do exist for the equations in the absence of a standard Lagrangian, we use an alternative way to construct the first integrals. We find the partial Noether operators and then first integrals by utilizing the partial Noether's theorem. This hopefully will give rise to further studies on first integrals for nonlinear systems from a partial Lagrangian viewpoint.

The outline of the paper is as follows. In section 2, the canonical form of a general linear system of two second-order ODEs is invoked and some basic definitions and results are adapted from the literature. Section 3 is related to partial Noether operators of a general linear system of two second-order ODEs that in general has no Lagrangian. The first integrals corresponding to partial Noether operators are given in section 4. Herein, even in the cases for which the linear system does not have a Lagrangian we obtain first integrals by the partial Noether theorem. The results of all cases are tabulated.

2. Preliminaries

Difficulties arise in the algebraic classification of general linear equations due to a huge number of arbitrary elements. Basically for a system of n second-order nonhomogeneous linear ODEs, $2n^2 + 2$ arbitrary elements exists. In general, the invertible transformations do not affect the number of symmetries, so one can obtain a simpler system (see [17]) before starting group classification, etc.

Consider a system of n second-order nonhomogeneous linear ODEs,

$$\mathbf{x}'' = A\mathbf{x}' + B\mathbf{x} + \mathbf{c}. \tag{1}$$

The system (1) can be mapped invertibly to one of the forms [17] given below,

$$\mathbf{y}'' = \bar{A}\mathbf{y}', \tag{2}$$

$$\mathbf{z}'' = \bar{B}\mathbf{z}, \tag{3}$$

where $A, B, \bar{A}, \bar{B}$ are $n \times n$ matrices and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{c}$ are vectors. The above theorem for the case $n = 2$ implies that the number of arbitrary elements reduces from $2 \times 2^2 + 2 = 10$ to 4.

Furthermore, any system of two linear second-order ODEs maps invertibly to the linear system [17],

$$\begin{cases} y'' = a(x)y + b(x)z, \\ z'' = c(x)y - a(x)z, \end{cases} \tag{4}$$

where a, b and c are arbitrary functions of x.

Now we present some definitions adapted from the literature [13, 15, 21, 22].

Definition 1. *Suppose we have*

$$N_\alpha(x, u, u', u'') = 0, \qquad \alpha = 1, 2 \tag{5}$$

which is a system of two second-order ODEs, where $u = (u^1, u^2) = (y, z)$ and $'$ is the derivative with respect to the independent variable x with $u' = \left(u^1_x, u^2_x\right) = (y', z')$. We assume that the system (5) can be expressed as

$$N_\alpha = N^0_\alpha + N^1_\alpha = 0, \qquad \alpha = 1, 2, \tag{6}$$

where N^0_α contain the second derivative terms and introduce multipliers $f^\alpha_\beta \in \mathcal{A}$ (note that $\mathcal{A}$ is the space of all differential functions) so that

$$f^\alpha_\beta N^0_\alpha + f^\alpha_\beta N^1_\alpha = 0, \qquad \beta = 1, 2 \tag{7}$$

(the matrix $\left(f^\alpha_\beta\right)$ is invertible). If there exists a function $L = L(x, u, u')$ such that (7) can be expressed as $\delta L/\delta u^\alpha = f^\beta_\alpha N^1_\beta$, then if $N^1_\beta \neq 0$ for some β, L is said to be a partial Lagrangian of the system (6). Otherwise it is known to be a standard Lagrangian.

If $N_\beta^\alpha \neq 0$ for some β, then

$$\frac{\delta L}{\delta u^\alpha} = f_\alpha^\beta N_\beta^1, \qquad \alpha = 1, 2, \tag{8}$$

are referred to as the partial Euler–Lagrange equations otherwise they are the Euler–Lagrange equations.

Definition 2. *Suppose that*

$$X = \xi(x, u)\frac{\partial}{\partial x} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} \tag{9}$$

is an operator in (x, u) *space, where* $u = (u^1, u^2) = (y, z)$ *is the dependent variable with coordinates* y *and* z*, and* x *is an independent variable. The generator* X *is said to be a partial Noether operator corresponding to a partial Lagrangian* $L(x, u, u')$ *of the system (8) if it can be determined from*

$$X^{[1]}L + (D_x\xi)L = \left(\eta^\alpha - \xi u_x^\alpha\right)\frac{\delta L}{\delta u^\alpha} + D_x(B), \tag{10}$$

with respect to some function $B(x, u)$ *with*

$$D_x = \frac{\partial}{\partial x} + u_x^\alpha\frac{\partial}{\partial u^\alpha} + u_{xx}^\alpha\frac{\partial}{\partial u_x^\alpha} + \cdots, \tag{11}$$

the total derivative operator. In (8)

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geqslant 1}(-D_x)^s\frac{\partial}{\partial u_s^\alpha}, \qquad \alpha = 1, 2, \tag{12}$$

where $u_1^\alpha \equiv u_x^\alpha$ *and* $u_2^\alpha \equiv u_{xx}^\alpha$*, etc, is the Euler operator and in (10)*

$$X^{[1]} = \xi\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial y} + \eta^2\frac{\partial}{\partial z} + \zeta_x^1\frac{\partial}{\partial y'} + \zeta_x^2\frac{\partial}{\partial z'} \tag{13}$$

the first prolongation of X*.*

Now we recall the Noether-like theorem from [15, 21].

Theorem (partial Noether's theorem)**.** *If the operator* X *in (9) is a partial Noether operator with respect to a partial Lagrangian* $L(x, u, u')$ *of (8), then the first integral of (8) can be constructed from the formula*

$$I = B - \left[\xi L + \left(\eta^\alpha - \xi u_x^\alpha\right)\frac{\partial L}{\partial u_x^\alpha}\right], \tag{14}$$

where B *is determined from (10).*

Remark. Equation (10) is the partial Noether determining equation provided $\delta L/\delta u^\alpha \neq 0$. If $\delta L/\delta u^\alpha = 0$, then we have a Lagrangian system and relation (10) reduces to the Noether determining equation as in Noether [1] (see also [2–4]).

3. Partial Noether operators

The operator X given in (9) is a partial Noether operator corresponding to a partial Lagrangian

$$L = \tfrac{1}{2}y'^2 + \tfrac{1}{2}z'^2, \tag{15}$$

with $\delta L/\delta y = -(a(x)y + b(x)z)$ and $\delta L/\delta z = a(x)z - c(x)y$ of (4) if it satisfies (10) which splits as

$$\xi_y = 0, \qquad \xi_z = 0, \tag{16}$$

$$\eta^1_y - \tfrac{1}{2}\xi_x = 0, \qquad \eta^2_z - \tfrac{1}{2}\xi_x = 0, \qquad \eta^1_z + \eta^2_y = 0, \tag{17}$$

$$\eta^1_x = (ay + bz)\xi + B_y, \tag{18}$$

$$\eta^2_x = (cy - az)\xi + B_z, \tag{19}$$

$$\eta^1(ay + bz) + \eta^2(cy - az) - B_x = 0. \tag{20}$$

Equations (16) and (17) result in

$$\xi = \alpha(x), \tag{21}$$

$$\eta^1 = \tfrac{1}{2}\alpha' y - C_1(x)z + A(x), \tag{22}$$

$$\eta^2 = \tfrac{1}{2}\alpha' z + C_1(x)y + C_2(x). \tag{23}$$

The replacement of the above equations in (18) and (19) gives

$$C_1'(x) = \tfrac{1}{2}(c - b)\alpha, \tag{24}$$

$$B = \tfrac{1}{4}\alpha'' y^2 - C_1'(x)yz + yA'(x) - \left(\tfrac{1}{2}ay^2 + byz\right)\alpha + S(x, z), \tag{25}$$

where

$$S(x, z) = \tfrac{1}{4}\alpha'' z^2 + C_2'(x)z + \tfrac{1}{2}az^2\alpha + C_3(x). \tag{26}$$

Equation (20) with the help of (24)–(26) reduces to the following system:

$$\frac{1}{4}\alpha''' + a\alpha' + \frac{1}{2}a'\alpha + \frac{1}{2}b\int (c - b)\alpha\,\mathrm{d}x + bA_1 = 0, \tag{27}$$

$$\frac{1}{2}b'\alpha + b\alpha' + \frac{1}{2}c'\alpha + c\alpha' - a\int (c - b)\alpha\,\mathrm{d}x - 2aA_1 = 0, \tag{28}$$

$$C_2''(x) + aC_2(x) = bA(x), \tag{29}$$

$$\frac{1}{4}\alpha''' - a\alpha' - \frac{1}{2}a'\alpha - \frac{1}{2}c\int (c - b)\alpha\,\mathrm{d}x - cA_1 = 0, \tag{30}$$

$$A''(x) - aA(x) = cC_2(x), \tag{31}$$

$$C_3'(x) = 0, \tag{32}$$

where A_1 is a constant.

From equation (32), we find that

$$C_3(x) = c_0. \tag{33}$$

In order to solve the system (27)–(31) and (24), the following cases need to be considered. We mention two of them and provide some details of the computations. All the cases though are listed in table 1.

Case 1. a, b and c are arbitrary

The following subcases of case 1 should be investigated.

Case 1.1. $b + c \neq 0$

In this case we easily find that

$$\alpha(x) = 0, \qquad C_1(x) = 0,$$
$$A(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x) + \alpha_3 u_3(x) + \alpha_4 u_4(x),$$
$$C_2(x) = \alpha_1 v_1(x) + \alpha_2 v_2(x) + \alpha_3 v_3(x) + \alpha_4 v_4(x).$$

The operators for this case can be written in the general form as

$$X_i = u_i \frac{\partial}{\partial y} + v_i \frac{\partial}{\partial z}, \qquad i = 1, \ldots, 4,$$

where (u_i, v_i) are linearly independent solutions of the adjoint of the system (4) and

$$B = yA'(x) + zC_2'(x).$$

Case 1.2. $b + c = 0$
The subcases of case 1.2 are

Case 1.2.1. $a \neq 0$
In this case we get the similar results as in case 1.1.

Case 1.2.2. $a = 0$
For this case we express the solution of the system (27)–(31) and (24) in the general form as

$$\alpha(x) = \sum_{i=1}^{4} \beta_i w_i(x), \qquad \beta_i \text{ constants}$$

$$C_1(x) = \frac{1}{4c} \sum_{i=1}^{4} \beta_i w_i'''(x),$$

$$A(x) = \sum_{i=5}^{8} \alpha_i u_i(x), \qquad \alpha_i \text{ constants}$$

$$C_2(x) = \sum_{i=5}^{8} \alpha_i v_i(x).$$

The operators and the guage terms are

$$X_i = w_i \frac{\partial}{\partial x} + \left(\frac{y}{2} w_i'(x) - \frac{z}{4c} w_i'''(x)\right) \frac{\partial}{\partial y} + \left(\frac{z}{2} w_i'(x) + \frac{y}{4c} w_i'''(x)\right) \frac{\partial}{\partial z}, \qquad i = 1, \ldots, 4,$$
$$X_j = u_j \frac{\partial}{\partial y} + v_j \frac{\partial}{\partial z}, \qquad j = 5, \ldots, 8,$$

where w_i represent the independent solutions of the resulting system of (27), (28) and (30) which are solutions of the linear system

$$\alpha^{(iv)} - \alpha''' \frac{c'}{c} - c^2 \alpha = 0, \qquad C_1(x) = \frac{1}{4c} \alpha'''$$

and (u_j, v_j) are the independent solutions of the adjoint of the system (4) with

$$B = \frac{1}{4}(y^2 + z^2)\alpha'' + y(-zC_1'(x) + A'(x)) - \left(\frac{ay^2}{2} - \frac{az^2}{2} + byz\right)\alpha + zC_2'(x).$$

Case 1.3. $a = x + s$, s is a constant
For this case we obtain similar results as in case 1.1.

The results for cases 2–4 were derived in [21] and are tabulated here in section 4. In [21], the case where a and b are arbitrary was given incorrectly. The simplest case 2.4 was also inadvertently omitted in [21] as well case 5. These are remedied in section 4.

Case 6. $a = a_0, b = b_0, c = c_0, a_0, b_0$ and c_0 are constants
Equations (27)–(31) become

$$\frac{1}{4}\alpha''' + a_0\alpha' + \frac{1}{2}b_0\int (c_0 - b_0)\alpha\,\mathrm{d}x + b_0A_1 = 0, \tag{34}$$

$$b_0\alpha' + c_0\alpha' - a_0\int (c_0 - b_0)\alpha\,\mathrm{d}x - 2a_0A_1 = 0, \tag{35}$$

$$C_2''(x) + a_0C_2(x) = b_0A(x), \tag{36}$$

$$\frac{1}{4}\alpha''' - a_0\alpha' - \frac{1}{2}c_0\int (c_0 - b_0)\alpha\,\mathrm{d}x - c_0A_1 = 0, \tag{37}$$

$$A''(x) - a_0A(x) = c_0C_2(x). \tag{38}$$

After some simple calculations, five subcases arise. We present the first one.

Case 6.1. $a_0 \neq 0, b_0 \neq 0$ and $c_0 \neq 0$
Whence the subcases of case 6.1 should be looked at. We just provide calculations relating to case 6.1.1.

Case 6.1.1. $b_0 - c_0 \neq 0$
The straightforward but lengthy calculations lead to

$$\begin{aligned}
&\alpha(x) = 0, \qquad C_1(x) = 0,\\
&C_2(x) = A_2\exp\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right) + A_3\exp\left(-\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\\
&\qquad + A_4\cos\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right) + A_5\sin\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right),\\
&A(x) = \frac{1}{b_0}\Big[A_2\left(\sqrt{a_0^2 + b_0c_0} + a_0\right)\exp\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\\
&\qquad + A_3\left(\sqrt{a_0^2 + b_0c_0} + a_0\right)\exp\left(-\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\\
&\qquad + A_4\left(-\sqrt{a_0^2 + b_0c_0} + a_0\right)\cos\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\\
&\qquad + A_5\left(-\sqrt{a_0^2 + b_0c_0} + a_0\right)\sin\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right).
\end{aligned} \tag{39}$$

The partial Noether operators and B in each case are constructed by choice of constant equal to one and the remaining constants equal to zero,

$$\begin{aligned}
&X_1 = \frac{1}{b_0}\left(\sqrt{a_0^2 + b_0c_0} + a_0\right)\exp\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\frac{\partial}{\partial y} + \exp\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\frac{\partial}{\partial z},\\
&B = \frac{y}{b_0}\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}\left(\sqrt{a_0^2 + b_0c_0} + a_0\right)\exp\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\\
&\qquad + z\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}\exp\left(\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right),\\
&X_2 = \frac{1}{b_0}\left(\sqrt{a_0^2 + b_0c_0} + a_0\right)\exp\left(-\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\frac{\partial}{\partial y} + \exp\left(-\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\frac{\partial}{\partial z},\\
&B = \frac{-y}{b_0}\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}\left(\sqrt{a_0^2 + b_0c_0} + a_0\right)\exp\left(-\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right)\\
&\qquad - z\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}\exp\left(-\left(a_0^2 + b_0c_0\right)^{\frac{1}{4}}x\right),
\end{aligned}$$

$$X_3 = \frac{1}{b_0}\left(-\sqrt{a_0^2+b_0c_0}+a_0\right)\cos\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x\frac{\partial}{\partial y}+\cos\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x\frac{\partial}{\partial z},$$

$$B = \frac{-y}{b_0}\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}\left(-\sqrt{a_0^2+b_0c_0}+a_0\right)\sin\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x - z\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}\sin\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x,$$

$$X_4 = \frac{1}{b_0}\left(-\sqrt{a_0^2+b_0c_0}+a_0\right)\sin\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x\frac{\partial}{\partial y}+\sin\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x\frac{\partial}{\partial z},$$

$$B = \frac{y}{b_0}\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}\left(-\sqrt{a_0^2+b_0c_0}+a_0\right)\cos\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x + z\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}\cos\left(a_0^2+b_0c_0\right)^{\frac{1}{4}}x. \tag{40}$$

These and all other cases of the partial Noether operators and guage terms are listed in table 1.

The interpretation of the results of all the cases are as follows:

Noether Operators:
The interpretation of the results for the special case $b(x) = c(x)$ of system (4) for which we do have a Lagrangian was provided in [21].

Partial Noether Operators:
Case 1.1: We find a four-dimensional Lie algebra in this case.
Case 1.2.1: We have a four-dimensional algebra.
Case 1.2.2: We deduce an eight-dimensional algebra which is distinct from the algebra of case 2.4.
Case 1.3: The Lie algebra for this case is four dimensional.
Case 6.1.1: The Lie algebra is four dimensional.
Case 6.1.2: In this case we get a five-dimensional Lie algebra.
Case 6.2.1: For this case the Lie algebra is four dimensional.
Case 6.2.2: We have a five-dimensional Lie algebra.
Case 6.3 and
Case 6.4: In cases 6.3 and 6.4, we obtain a four-dimensional Lie algebra.
Case 6.5: The Lie algebra is eight dimensional.
Case 7.1–7.4: The Lie algebra in each of cases 7.1–7.4 is four dimensional too.
Case 8.1–8.4: We also obtain a four-dimensional Lie algebra in cases 8.1–8.4.

Case 9: In each of the subcases of case 9, we deduce a four-dimensional Lie algebra.
Case 10.1–10.4: The Lie algebra for each of cases 10.1–10.4 is four dimensional.

These form subalgebras of the Lie algebras of the linear system studied in [17] since $\delta L/\delta y = -(a(x)y + b(x)z)$ and $\delta L/\delta z = a(x)z - c(x)y$, which means that these are independent of derivatives and the partial Noether operators are symmetry generators [15] of the Euler–Lagrange equations. Note that in general if the partial Euler–Lagrange equations are free of derivatives then the partial Noether operators become symmetry generators of the equations and the Lie algebras for both are isomorphic.

As the system under consideration does not have a standard Lagrangian, the partial Lagrangian approach is very useful in constructing first integrals for such equations. Partial Lagrangians do exist for second-order equations in the absence of standard Lagrangians.

In the following section, we derive the first integrals for each of cases 1–10.

Table 1. Partial Noether operators, guage terms and first integrals of system (4).

Case 1: a, b and c are arbitrary

Case 1.1: $b + c \neq 0$
$X_i = u_i \frac{\partial}{\partial y} + v_i \frac{\partial}{\partial z}, i = 1, \ldots, 4, B = yA'(x) + zC_2'(x)$
$I_i = yA'(x) + zC_2'(x) - u_i y' - v_i z', i = 1, \ldots, 4$
(u_i, v_i) are linearly independent solutions of the adjoint of the system (4)

Case 1.2: $b + c = 0$

Case 1.2.1: $a \neq 0$
In this case we obtain similar results as in case 1.1

Case 1.2.2: $a = 0$
$X_i = w_i \frac{\partial}{\partial x} + (\frac{y}{2} w_i'(x) - \frac{z}{4c} w_i'''(x)) \frac{\partial}{\partial y} + (\frac{z}{2} w_i'(x) + \frac{y}{4c} w_i'''(x)) \frac{\partial}{\partial z}$
$B = \frac{1}{4}(y^2 + z^2) w_i'' + y[-\frac{z}{4c} w_i'''' + \frac{z}{4c^2} w_i'''] - (\frac{ay^2}{2} - \frac{az^2}{2} + byz) w_i$
$I_i = \frac{1}{4}(y^2 + z^2) w_i'' + y[-\frac{z}{4c} w_i'''' + \frac{z}{4c^2} w_i'''] - (\frac{ay^2}{2} - \frac{az^2}{2} + byz) w_i + \frac{1}{2}(y'^2 + z'^2) w_i$
$-[\frac{1}{2}(yy' + zz') w_i' + \frac{1}{4c}(y'z - yz') w_i''', i = 1, \ldots, 4$
$X_j = u_j \frac{\partial}{\partial y} + v_j \frac{\partial}{\partial z}, B = yu_j' + zv_j'$
$I_j = yu_j' + zv_j' - u_j y' - v_j z', j = 5, \ldots, 8$

Case 1.3: $a = x + s$, s is a constant
The results for this case are similar to case 1.1
For the special case when $b(x) = c(x)$ with a and b arbitrary we get the same results as given above for case 1.1

The results for cases 2–5 also belong to the special case $b(x) = c(x)$ for which a Lagrangian exists. In this case system (4) is self-adjoint

Case 2: $a = a_0, b = b_0, a_0$ and b_0 are constants

Case 2.1: If $a_0 \neq 0, b_0 \neq 0$,
$X_1 = \frac{\partial}{\partial x}, B = 0, I_1 = \frac{1}{2} y'^2 + \frac{1}{2} z'^2 + \frac{1}{2} a_0 z^2 - \frac{1}{2} a_0 y^2 - b_0 yz$

$X_2 = \frac{1}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x) \frac{\partial}{\partial y} + \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x) \frac{\partial}{\partial z},$
$B = \frac{y}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0)(a_0^2 + b_0^2)^{\frac{1}{4}} \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x) + z(a_0^2 + b_0^2)^{\frac{1}{4}} \exp(a_0^2 + b_0^2)^{\frac{1}{4}} x$
$I_2 = \exp((a_0^2 + b_0^2)^{\frac{1}{4}} x)$
$\quad \times [\frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0^2} + a_0) + (a_0^2 + b_0^2)^{\frac{1}{4}} z - \frac{y'}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) - z']$

$X_3 = \frac{1}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x) \frac{\partial}{\partial y} + \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x) \frac{\partial}{\partial z}$
$B = \frac{-y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0^2} + a_0) \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x) - z(a_0^2 + b_0^2)^{\frac{1}{4}} \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x)$
$I_3 = \exp(-(a_0^2 + b_0^2)^{\frac{1}{4}} x)$
$\quad \times [-\frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(\sqrt{a_0^2 + b_0^2} + a_0) - (a_0^2 + b_0^2)^{\frac{1}{4}} z - \frac{y'}{b_0}(\sqrt{a_0^2 + b_0^2} + a_0) - z']$

$X_4 = \frac{1}{b_0}(-\sqrt{a_0^2 + b_0^2} + a_0) \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x \frac{\partial}{\partial y} + \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x \frac{\partial}{\partial z}$
$B = \frac{-y}{b_0}(-\sqrt{a_0^2 + b_0^2} + a_0)(a_0^2 + b_0^2)^{\frac{1}{4}} \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x - z(a_0^2 + b_0^2)^{\frac{1}{4}} \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x$
$I_4 = -\frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}} \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x(-\sqrt{a_0^2 + b_0^2} + a_0) - z(a_0^2 + b_0^2)^{\frac{1}{4}} \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x$
$\quad - [\frac{y'}{b_0} \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x(-\sqrt{a_0^2 + b_0^2} + a_0) + z' \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x]$

$X_5 = \frac{1}{b_0}(-\sqrt{a_0^2 + b_0^2} + a_0) \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x \frac{\partial}{\partial y} + \sin(a_0^2 + b_0^2)^{\frac{1}{4}} x \frac{\partial}{\partial z}$
$B = \frac{y}{b_0}(a_0^2 + b_0^2)^{\frac{1}{4}}(-\sqrt{a_0^2 + b_0^2} + a_0) \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x + z(a_0^2 + b_0^2)^{\frac{1}{4}} \cos(a_0^2 + b_0^2)^{\frac{1}{4}} x$

Table 1. (Continued.)

$I_5 = \frac{y}{b_0}(a_0^2+b_0^2)^{\frac{1}{4}}\cos(a_0^2+b_0^2)^{\frac{1}{4}}x(-\sqrt{a_0^2+b_0^2}+a_0) + z(a_0^2+b_0^2)^{\frac{1}{4}}\cos(a_0^2+b_0^2)^{\frac{1}{4}}x$
$\quad - [\frac{y'}{b_0}\sin(a_0^2+b_0^2)^{\frac{1}{4}}x(-\sqrt{a_0^2+b_0^2}+a_0) + z'\sin(a_0^2+b_0^2)^{\frac{1}{4}}x]$

Case 2.2: If $a_0 = 0, b_0 \neq 0$
$X_1 = \frac{\partial}{\partial x}, B = 0, I_1 = \frac{1}{2}y'^2 + \frac{1}{2}z'^2 - b_0 yz$
$X_2 = \exp(\sqrt{b_0}x)\frac{\partial}{\partial y} + \exp(\sqrt{b_0}x)\frac{\partial}{\partial z}, B = y\sqrt{b_0}\exp(\sqrt{b_0}x) + z\sqrt{b_0}\exp(\sqrt{b_0}x)$
$I_2 = \exp(\sqrt{b_0}x)[y\sqrt{b_0} + \sqrt{b_0}z - y' - z']$
$X_3 = \exp(-\sqrt{b_0}x)\frac{\partial}{\partial y} + \exp(-\sqrt{b_0}x)\frac{\partial}{\partial z}, B = -y\sqrt{b_0}\exp(-\sqrt{b_0}x) - z\sqrt{b_0}\exp(-\sqrt{b_0}x)$
$I_3 = \exp(-\sqrt{b_0}x)[-y\sqrt{b_0} - \sqrt{b_0}z - y' - z']$
$X_4 = -\cos\sqrt{b_0}x\frac{\partial}{\partial y} + \cos\sqrt{b_0}x\frac{\partial}{\partial z}, B = y\sqrt{b_0}\sin\sqrt{b_0}x - z\sqrt{b_0}\sin\sqrt{b_0}x$
$I_4 = y\sqrt{b_0}\sin\sqrt{b_0}x - z\sqrt{b_0}\sin\sqrt{b_0}x + y'\cos\sqrt{b_0}x - z'\cos\sqrt{b_0}x$
$X_5 = -\sin\sqrt{b_0}x\frac{\partial}{\partial y} + \sin\sqrt{b_0}x\frac{\partial}{\partial z}, B = -y\sqrt{b_0}\cos\sqrt{b_0}x + z\sqrt{b_0}\cos\sqrt{b_0}x$
$I_5 = -y\sqrt{b_0}\cos\sqrt{b_0}x + z\sqrt{b_0}\cos\sqrt{b_0}x + y'\sin\sqrt{b_0}x - z'\sin\sqrt{b_0}x$

Case 2.3: $a_0 \neq 0, b_0 = 0$
$X_1 = \frac{\partial}{\partial x}, B = 0, I_1 = -\frac{1}{2}a_0y^2 + \frac{1}{2}a_0z^2 + \frac{1}{2}y'^2 + \frac{1}{2}z'^2$
$X_2 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y}, B = y\sqrt{a_0}\exp(\sqrt{a_0}x), I_2 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x)$
$X_3 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y}, B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x), I_3 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x)$
$X_4 = \cos\sqrt{a_0}x\frac{\partial}{\partial z}, B = -z\sqrt{a_0}\sin\sqrt{a_0}x, I_4 = -z\sqrt{a_0}\sin(\sqrt{a_0}x) - z'\cos(\sqrt{a_0}x)$
$X_5 = \sin\sqrt{a_0}x\frac{\partial}{\partial z}, B = z\sqrt{a_0}\cos(\sqrt{a_0}x), I_5 = z\sqrt{a_0}\cos(\sqrt{a_0}x) - z'\sin(\sqrt{a_0}x)$

Case 2.4: $a_0 = 0, b_0 = 0$
$X_1 = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, B = 0, I_1 = y'z - yz'$
$X_2 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} + xz\frac{\partial}{\partial z}, B = \frac{1}{2}(y^2+z^2), I_2 = \frac{1}{2}(y^2+z^2) + \frac{x^2}{2}(y'^2+z'^2) - x(yy'+zz')$
$X_3 = x\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} + \frac{z}{2}\frac{\partial}{\partial z}, B = 0, I_3 = \frac{x}{2}(y'^2+z'^2) - \frac{1}{2}(yy'+zz')$
$X_4 = \frac{\partial}{\partial x}, B = 0, I_4 = \frac{1}{2}(y'^2+z'^2), X_5 = x\frac{\partial}{\partial y}, B = y, I_5 = y - xy'$
$X_6 = \frac{\partial}{\partial y}, B = 0, I_6 = -y', X_7 = x\frac{\partial}{\partial z}, B = z, I_7 = z - xz', X_8 = \frac{\partial}{\partial z}, B = 0, I_8 = -z'$

Case 3: $a = a_0, b \neq$ constant, a_0 is a constant

Case 3.1: If $a_0 \neq 0$ and $b \neq$ constant
In this case we get the similar operators and first integrals as given in case 1.1

Case 3.2: $a_0 = 0, b \neq$ constant
$X_i = \alpha(x)\frac{\partial}{\partial x} + u_i\frac{\partial}{\partial y} + v_i\frac{\partial}{\partial z}, B = \frac{1}{2}(y^2+z^2)A_4 + yA'(x) + zC_2'(x)$
$I_i = \frac{1}{2}(y^2+z^2)A_4 + yA'(x) + zC_2'(x) + \frac{1}{2}y'^2\alpha + \frac{1}{2}z'^2\alpha - byz\alpha - u_iy' - v_iz', i = 1, \ldots, 4$
(u_i, v_i) are linearly independent solutions of the adjoint of the system (4)

Case 4: $a \neq$ constant, $b = b_0$, b_0=constant

Case 4.1: $a \neq$ constant, $b_0 \neq 0$
For this case we get similar generators, B and first integrals as given in case 1.1

Case 4.2: $a \neq$ constant, $b_0 = 0$
The generators, B and first integrals in this case are similar to those obtained in case 3.2

Case 5: $a = \lambda b, \lambda =$ constant
For this case the results agree with those given in case 1.1

Case 6: $a = a_0, b = b_0, c = c_0, a_0, b_0$ and c_0 are constants

Case 6.1: $a_0 \neq 0, b_0 \neq 0$ and $c_0 \neq 0$

Case 6.1.1: $b_0 - c_0 \neq 0$
$X_1 = \frac{1}{b_0}(\sqrt{a_0^2+b_0c_0} + a_0)\exp((a_0^2+b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp((a_0^2+b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z}$

Table 1. (Continued.)

$B = \frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(\sqrt{a_0^2+b_0c_0}+a_0)\exp((a_0^2+b_0c_0)^{\frac{1}{4}}x) + z(a_0^2+b_0c_0)^{\frac{1}{4}}\exp((a_0^2+b_0c_0)^{\frac{1}{4}}x)$

$I_1 = \exp((a_0^2+b_0c_0)^{\frac{1}{4}}x)[\frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(\sqrt{a_0^2+b_0c_0}+a_0) + z(a_0^2+b_0c_0)^{\frac{1}{4}}$
$\quad - \frac{y'}{b_0}(\sqrt{a_0^2+b_0c_0}+a_0) - z']$

$X_2 = \frac{1}{b_0}(\sqrt{a_0^2+b_0c_0}+a_0)\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z}$

$B = \frac{-y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(\sqrt{a_0^2+b_0c_0}+a_0)\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)$
$\quad - z(a_0^2+b_0c_0)^{\frac{1}{4}}\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)$

$I_2 = -\exp(-(a_0^2+b_0c_0)^{\frac{1}{4}}x)[\frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(\sqrt{a_0^2+b_0c_0}+a_0) + z(a_0^2+b_0c_0)^{\frac{1}{4}}$
$\quad + \frac{y'}{b_0}(\sqrt{a_0^2+b_0c_0}+a_0) + z']$

$X_3 = \frac{1}{b_0}(-\sqrt{a_0^2+b_0c_0}+a_0)\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial y} + \cos(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial z}$

$B = \frac{-y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0c_0}+a_0)\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x$
$\quad - z(a_0^2+b_0c_0)^{\frac{1}{4}}\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x$

$I_3 = -\frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0c_0}+a_0)\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x - z(a_0^2+b_0c_0)^{\frac{1}{4}}\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x$
$\quad - [\frac{y'}{b_0}(-\sqrt{a_0^2+b_0c_0}+a_0)\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x + z'\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x]$

$X_4 = \frac{1}{b_0}(-\sqrt{a_0^2+b_0c_0}+a_0)\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial y} + \sin(a_0^2+b_0c_0)^{\frac{1}{4}}x\frac{\partial}{\partial z}$

$B = \frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0c_0}+a_0)\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x + z(a_0^2+b_0c_0)^{\frac{1}{4}}\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x$

$I_4 = \frac{y}{b_0}(a_0^2+b_0c_0)^{\frac{1}{4}}(-\sqrt{a_0^2+b_0c_0}+a_0)\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x + z(a_0^2+b_0c_0)^{\frac{1}{4}}\cos(a_0^2+b_0c_0)^{\frac{1}{4}}x$
$-[\frac{y'}{b_0}(-\sqrt{a_0^2+b_0c_0}+a_0)\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x + z'\sin(a_0^2+b_0c_0)^{\frac{1}{4}}x]$

Case 6.1.2: $b_0 - c_0 = 0$
This reduces to case 2.1

Case 6.2: $a_0 = 0, b_0 \neq 0, c_0 \neq 0$

Case 6.2.1: $b_0 - c_0 \neq 0$

$X_1 = \sqrt{\frac{c_0}{b_0}}\exp((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z}$

$B = \frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\exp((b_0c_0)^{\frac{1}{4}}x) + z(b_0c_0)^{\frac{1}{4}}\exp((b_0c_0)^{\frac{1}{4}}x)$

$I_1 = \exp((b_0c_0)^{\frac{1}{4}}x)[\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} + z(b_0c_0)^{\frac{1}{4}} - y'\sqrt{\frac{c_0}{b_0}} - z']$

$X_2 = \sqrt{\frac{c_0}{b_0}}\exp(-(b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \exp(-(b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z}$

$B = -\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\exp(-(b_0c_0)^{\frac{1}{4}}x) - z(b_0c_0)^{\frac{1}{4}}\exp(-(b_0c_0)^{\frac{1}{4}}x)$

$I_2 = -\exp(-(b_0c_0)^{\frac{1}{4}}x)[\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}} + z(b_0c_0)^{\frac{1}{4}} + y'\sqrt{\frac{c_0}{b_0}} + z']$

$X_3 = -\sqrt{\frac{c_0}{b_0}}\cos((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \cos((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z}$

$B = \frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\sin(b_0c_0)^{\frac{1}{4}}x - z(b_0c_0)^{\frac{1}{4}}\sin(b_0c_0)^{\frac{1}{4}}x$

$I_3 = \frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\sin(b_0c_0)^{\frac{1}{4}}x - z(b_0c_0)^{\frac{1}{4}}\sin(b_0c_0)^{\frac{1}{4}}x + y'\sqrt{\frac{c_0}{b_0}}\cos(b_0c_0)^{\frac{1}{4}}x - z'\cos(b_0c_0)^{\frac{1}{4}}x$

$X_4 = -\sqrt{\frac{c_0}{b_0}}\sin((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial y} + \sin((b_0c_0)^{\frac{1}{4}}x)\frac{\partial}{\partial z}$

$B = -\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\cos(b_0c_0)^{\frac{1}{4}}x + z(b_0c_0)^{\frac{1}{4}}\cos(b_0c_0)^{\frac{1}{4}}x$

$I_4 = -\frac{y}{b_0}(b_0c_0)^{\frac{3}{4}}\cos(b_0c_0)^{\frac{1}{4}}x + z(b_0c_0)^{\frac{1}{4}}\cos(b_0c_0)^{\frac{1}{4}}x + y'\sqrt{\frac{c_0}{b_0}}\sin(b_0c_0)^{\frac{1}{4}}x - z'\sin(b_0c_0)^{\frac{1}{4}}x$

Case 6.2.2: $b_0 - c_0 = 0$
The results for this case are identical to case 2.2

Table 1. (Continued.)

Case 6.3: $a_0 \neq 0, b_0 = 0, c_0 \neq 0$
$X_1 = -\frac{c_0}{2a_0}\cos(\sqrt{a_0}x)\frac{\partial}{\partial y} + \cos(\sqrt{a_0}x)\frac{\partial}{\partial z}$, $B = \frac{c_0}{2\sqrt{a_0}} y\sin(\sqrt{a_0}x) - z\sqrt{a_0}\sin(\sqrt{a_0}x)$
$I_1 = \frac{c_0}{2\sqrt{a_0}} y\sin\sqrt{a_0}x - z\sqrt{a_0}\sin\sqrt{a_0}x + \frac{c_0}{2a_0}y'\cos\sqrt{a_0}x - z'\cos\sqrt{a_0}x$
$X_2 = -\frac{c_0}{2a_0}\sin(\sqrt{a_0}x)\frac{\partial}{\partial y} + \sin(\sqrt{a_0}x)\frac{\partial}{\partial z}$, $B = -\frac{c_0}{2\sqrt{a_0}} y\cos(\sqrt{a_0}x) + z\sqrt{a_0}\cos(\sqrt{a_0}x)$
$I_2 = -\frac{c_0}{2\sqrt{a_0}} y\cos\sqrt{a_0}x + z\sqrt{a_0}\cos\sqrt{a_0}x + \frac{c_0}{2a_0}y'\sin\sqrt{a_0}x - z'\sin\sqrt{a_0}x$
$X_3 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y}$, $B = y\sqrt{a_0}\exp(\sqrt{a_0}x)$, $I_3 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x)$
$X_4 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y}$, $B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x)$, $I_4 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x)$
If $a_0 < 0$, one obtains exponential solutions of (29) and trigonometric solutions of (31)

Case 6.4: $a_0 \neq 0, b_0 \neq 0, c_0 = 0$
$X_1 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y} + \frac{b_0}{2a_0}\exp(\sqrt{a_0}x)\frac{\partial}{\partial z}$, $B = y\sqrt{a_0}\exp(\sqrt{a_0}x) + z\frac{b_0}{2\sqrt{a_0}}\exp(\sqrt{a_0}x)$
$I_1 = \exp(\sqrt{a_0}x)[y\sqrt{a_0} + \frac{b_0}{2\sqrt{a_0}}z - y' - \frac{b_0}{2a_0}z']$
$X_2 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y} + \frac{b_0}{2a_0}\exp(-\sqrt{a_0}x)\frac{\partial}{\partial z}$, $B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - z\frac{b_0}{2\sqrt{a_0}}\exp(-\sqrt{a_0}x)$
$I_2 = -\exp(-\sqrt{a_0}x)[y\sqrt{a_0} + \frac{b_0}{2\sqrt{a_0}}z + y' + \frac{b_0}{2a_0}z']$
$X_3 = \cos\sqrt{a_0}x\frac{\partial}{\partial z}$, $B = -z\sqrt{a_0}\sin(\sqrt{a_0}x)$, $I_3 = -z\sqrt{a_0}\sin\sqrt{a_0}x - z'\cos\sqrt{a_0}x$
$X_4 = \sin\sqrt{a_0}x\frac{\partial}{\partial z}$, $B = z\sqrt{a_0}\cos(\sqrt{a_0}x)$, $I_4 = z\sqrt{a_0}\cos\sqrt{a_0}x - z'\sin\sqrt{a_0}x$
If $a_0 < 0$, one finds trigonometric solutions of (31) and exponential solutions of (29)

Case 6.5: $a_0 = 0, b_0 = 0, c_0 = 0$
The results for this case are the same as case 2.4

Case 7: $a = a_0, b = b_0, c \neq$ constant, a_0 and b_0 are constants

Case 7.1: $a_0 \neq 0, b_0 \neq 0, c \neq$ constant

Case 7.2: $a_0 = 0, b_0 \neq 0, c \neq$ constant
For cases 7.1 and 7.2 we get similar operators and first integrals as given in case 1.1

Case 7.3: $a_0 \neq 0, b_0 = 0, c \neq$ constant
$X_1 = u_1(x)\frac{\partial}{\partial y} + \cos\sqrt{a_0}x\frac{\partial}{\partial z}$, $B = yu_1'(x) - z\sqrt{a_0}\sin\sqrt{a_0}x$
$I_1 = yu_1'(x) - y'u_1(x) - z\sqrt{a_0}\sin\sqrt{a_0}x - z'\cos\sqrt{a_0}x$

$X_2 = u_2(x)\frac{\partial}{\partial y} + \sin\sqrt{a_0}x\frac{\partial}{\partial z}$, $B = yu_2'(x) + z\sqrt{a_0}\cos\sqrt{a_0}x$
$I_2 = yu_2'(x) - y'u_2(x) + z\sqrt{a_0}\cos\sqrt{a_0}x - z'\sin\sqrt{a_0}x$
$X_3 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y}$, $B = y\sqrt{a_0}\exp(\sqrt{a_0}x)$, $I_3 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x)$
$X_4 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y}$, $B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x)$
$I_4 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x)$
u_i are linearly independent solutions of (31). If $a_0 < 0$, then one gets exponential functions for (29)

Case 7.4: $a_0 = 0, b_0 = 0, c \neq$ constant
$X_1 = u_1(x)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$, $B = yu_1'(x)$, $I_1 = yu_1'(x) - y'u_1(x) - z'$
$X_2 = u_2(x)\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$, $B = yu_2'(x) + z$, $I_2 = yu_2'(x) - y'u_2(x) + z - xz'$
$X_3 = \frac{\partial}{\partial y}$, $B = 0$, $I_3 = -y'$
$X_4 = x\frac{\partial}{\partial y}$, $B = y$, $I_4 = y - xy'$
u_i are the independent solutions of (31) given by $u_1 = \int\int c(x)\,dx\,dx$, $u_2 = \int\int xc(x)\,dx\,dx$

Case 8: $a = a_0, b \neq$ constant, $c = c_0$, a_0 and c_0 are constants

Case 8.1: $a_0 \neq 0, b \neq$ constant, $c_0 \neq 0$

Case 8.2: $a_0 = 0, b \neq$ constant, $c_0 \neq 0$
For cases 8.1 and 8.2, the results are similar to case 1.1

Table 1. (Continued.)

Case 8.3: $a_0 \neq 0$, $b \neq$ constant, $c_0 = 0$
$X_1 = \exp(\sqrt{a_0}x)\frac{\partial}{\partial y} + v_1(x)\frac{\partial}{\partial z}$, $B = y\sqrt{a_0}\exp(\sqrt{a_0}x) + zv_1'(x)$
$I_1 = y\sqrt{a_0}\exp(\sqrt{a_0}x) - y'\exp(\sqrt{a_0}x) + zv_1'(x) - z'v_1(x)$

$X_2 = \exp(-\sqrt{a_0}x)\frac{\partial}{\partial y} + v_2(x)\frac{\partial}{\partial z}$, $B = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) + zv_2'(x)$
$I_2 = -y\sqrt{a_0}\exp(-\sqrt{a_0}x) - y'\exp(-\sqrt{a_0}x) + zv_2'(x) - z'v_2(x)$

$X_3 = \cos(\sqrt{a_0}x)\frac{\partial}{\partial z}$, $B = -z\sqrt{a_0}\sin\sqrt{a_0}x$, $I_3 = -z\sqrt{a_0}\sin\sqrt{a_0}x - z'\cos\sqrt{a_0}x$

$X_4 = \sin(\sqrt{a_0}x)\frac{\partial}{\partial z}$, $B = z\sqrt{a_0}\cos\sqrt{a_0}x$, $I_2 = z\sqrt{a_0}\cos\sqrt{a_0}x - z'\sin\sqrt{a_0}x$
v_i are the particular solutions of (29). If $a_0 < 0$, then one obtains trigonometric solutions of (31) and exponential solutions of (29)

Case 8.4: $a_0 = 0$, $b \neq$ constant, $c_0 = 0$
$X_1 = \frac{\partial}{\partial y} + v_1(x)\frac{\partial}{\partial z}$, $B = zv_1'(x)$, $I_1 = zv_1'(x) - z'v_1(x) - y'$
$X_2 = x\frac{\partial}{\partial y} + v_2(x)\frac{\partial}{\partial z}$, $B = y + zv_2'(x)$, $I_2 = zv_2'(x) - z'v_2(x) + y - xy'$
$X_3 = \frac{\partial}{\partial z}$, $B = 0$, $I_3 = -z'$
$X_4 = x\frac{\partial}{\partial z}$, $B = z$, $I_4 = z - xz'$
v_i are the particular solutions of (29)

Case 9: $a \neq$ constant, $b = b_0$, $c = c_0$, b_0 and c_0 are constants
Case 9.1: $a \neq$ constant, $b_0 \neq 0$, $c_0 \neq 0$
Case 9.1.1: $b_0 + c_0 \neq 0$
Case 9.1.2: $b_0 + c_0 = 0$
Case 9.2: $a \neq$ constant, $b_0 = 0$, $c_0 \neq 0$
Case 9.3: $a \neq$ constant, $b_0 \neq 0$, $c_0 = 0$
Case 9.4: $a \neq$ constant, $b_0 = 0$, $c_0 = 0$
For all subcases of case 9, we get similar operators and first integrals as given in case 1.1

Case 10: a, b and c are related to each other and $a \neq 0$, $b \neq 0$, $c \neq 0$
Case 10.1: $a = \lambda_1 b$, $b = \lambda_2 c$, λ_1, λ_2 are constants
Case 10.2: $a = \lambda b$, c is arbitrary, $\lambda =$ constant
Case 10.3: $a = \lambda c$, b is arbitrary, $\lambda =$ constant
Case 10.4: $b = \lambda c$, a is arbitrary, $\lambda =$ constant
For cases 10.1–10.4 we obtain similar results as in case 1.1

4. First integrals

If the operator X in (9) is a partial Noether operator corresponding to the partial Lagrangian L given in (15) of the general linear system of two second-order ODEs (4), then the first integrals are found from the formula (14).

The first integrals for each case are summarized in table 1. The partial Noether operators are also presented for completeness.

5. Applications to physical systems

We consider some examples of physical systems of two linear nonhomogeneous second-order ODEs including ones that do not admit standard Lagrangians. We utilize transformations that reduce systems of two linear second-order ODEs into the canonical form (3). Then any system of two linear second-order ODEs can be mapped to the linear system (4) (see [17]).

We employ the linear change of variables [23],

$$y = Mz + y^*, \qquad M = [m_{ij}(x)], \tag{41}$$

where $y^* = (y, z)^T$ is a particular solution of (1) and $m_j = (m_{ij})^T, i, j = 1, 2$ are two linearly independent solutions of $2y' - Ay = 0$. Under (41) the system (1) is transformed to the canonical form (3) in which $\bar{B} = M^{-1}(AM' + BM - M'')$.

(1) In the first example we consider the conservative system with two degrees of freedom (see [27])

$$y'' = 3y - 2z, \qquad z'' = -y + 2z. \tag{42}$$

It can be easily seen that the system (42) is in canonical form (3).
Applying the change of variables (see [17])

$$\bar{y} = y/\phi(x), \qquad \bar{z} = z/\phi(x), \qquad \bar{x} = \int^x \phi^{-2}(s)\,\mathrm{d}s, \tag{43}$$

where ϕ satisfies

$$\phi'' - \tfrac{5}{2}\phi = 0, \tag{44}$$

which results in

$$\phi = c_1 \exp\left(\sqrt{\tfrac{5}{2}}x\right) + c_2 \exp\left(-\sqrt{\tfrac{5}{2}}x\right), \qquad c_i = \text{constants}, \tag{45}$$

we find that the system (42) becomes

$$\bar{y}'' = \frac{\phi^4}{2}\bar{y} - 2\phi^4\bar{z}, \qquad \bar{z}'' = -\phi^4\bar{y} - \frac{\phi^4}{2}\bar{z}. \tag{46}$$

Note that the system (46) belongs to case 10.1 and we construct four first integrals as listed in the table. In [27] three first integrals were reported.

(2) The time-dependent oscillator system (note that x is taken as the time here),

$$y'' + \omega_1^2(x)y = 0, \qquad z'' + \omega_2^2(x)z = 0, \tag{47}$$

where $\omega_1(x)$ and $\omega_2(x)$ are the frequencies, is investigated. We are interested in finding the canonical form of system (47). Simple inspection shows that system (47) associated with equation (1) is already in canonical form (3). The system (47) will be reduced to the following form under the change of variables as in (43):

$$\bar{y}'' = -\nu(x)\bar{y}, \qquad \bar{z}'' = \nu(x)\bar{z}, \tag{48}$$

where

$$\nu(x) = \left(\omega_1^2(x) - \omega_2^2(x)\right)\frac{\phi^4}{2}, \tag{49}$$

in which ϕ is the solution of the one-dimensional time-dependent oscillator

$$\phi'' + \frac{\omega_1^2 + \omega_2^2}{2}\phi = 0. \tag{50}$$

At this point we consider the following assumptions.

(2.1) If $\omega_1 = \omega_2$ (resonant case), then system (48) transforms to

$$\bar{y}'' = 0, \qquad \bar{z}'' = 0, \tag{51}$$

which corresponds to case 6.5 and we have computed eight first integrals for this cases given in the table.

(2.2) If $\omega_1 \neq \omega_2$ (non-resonant case) then system (48) falls into case 9.4 which results in four integrals as mentioned in the table.

(3) A linearly damped vibrating system with two degrees of freedom can be governed by the equations of motion [27]

$$y'' = -\omega^2 y - a_{11}y' - a_{12}z', \qquad z'' = -\Omega^2 z - a_{21}y' - a_{22}z', \tag{52}$$

in which ω^2, Ω^2 and a_{ij} are constant parameters.

Perform the linear change of variables (41) and for the sake of simplicity we choose $a_{11} = -4, a_{12} = -6, a_{21} = -4, a_{22} = -2$. Simple manipulations show that

$$M = \begin{pmatrix} \exp(-x) & 3\exp(4x) \\ -\exp(-x) & 2\exp(4x) \end{pmatrix}$$

is the solution of $2y' - Ay = 0$ and the system (52) takes the form given in (3). One can easily check that

$$\bar{B} = \begin{pmatrix} \frac{2}{5}\omega^2 - \frac{3}{5}\Omega^2 + 1 & \frac{6}{5}(\omega^2 + \Omega^2)\exp(5x) \\ \frac{1}{5}(\omega^2 + \Omega^2)\exp(-5x) & \frac{3}{5}\omega^2 - \frac{2}{5}\Omega^2 + 16 \end{pmatrix}.$$

The resulting system can also be expressed as

$$y'' = ay + b(x)z, \qquad z'' = c(x)y + dz, \tag{53}$$

where

$$\begin{aligned} a &= \tfrac{2}{5}\omega^2 - \tfrac{3}{5}\Omega^2 + 1, & b(x) &= \tfrac{6}{5}(\omega^2 + \Omega^2)\exp(5x), \\ c(x) &= \tfrac{1}{5}(\omega^2 + \Omega^2)\exp(-5x), & d &= \tfrac{3}{5}\omega^2 - \tfrac{2}{5}\Omega^2 + 16. \end{aligned} \tag{54}$$

The system (53) reduces to the following by using the transformations (43):

$$\bar{y}'' = \alpha(\bar{x})\bar{y} + \beta(\bar{x})\bar{z}, \qquad \bar{z}'' = \gamma(\bar{x})\bar{y} - \alpha(\bar{x})\bar{z}, \tag{55}$$

where

$$\bar{x} = \frac{\exp(2\sqrt{7}x)}{2\sqrt{7}}, \qquad \alpha = \frac{\phi^4(a-d)}{2}, \qquad \beta = \phi^4 b(x), \qquad \gamma = \phi^4 c(x). \tag{56}$$

In (56) ϕ satisfies

$$\phi'' - \frac{a+d}{2}\phi = 0. \tag{57}$$

If we choose $\omega = 1, \Omega = 2$, then from (57) $\phi = c_3\exp(\sqrt{7}x) + c_4\exp(-\sqrt{7}x)$ or we can select $\phi = \exp(-\sqrt{7}x)$ for $c_3 = 0$ and $c_4 = 1$. Equation (56) finally results in

$$\alpha = -8[2\sqrt{7}\bar{x}]^{-2}, \qquad \beta = 6[2\sqrt{7}\bar{x}]^{\frac{5-4\sqrt{7}}{2\sqrt{7}}}, \qquad \gamma = [2\sqrt{7}\bar{x}]^{\frac{-(5+4\sqrt{7})}{2\sqrt{7}}}. \tag{58}$$

The system (55) in comparison with case 1.1 gives four first integrals. In [28], the case $\omega = \Omega$ was considered but only two integrals were reported.

6. Conclusion

The first integrals for the general linear system of two second-order ODEs with variable coefficients are derived by using a partial Lagrangian approach. In general, the underlying system of two equations does not have a standard Lagrangian which can be verified from Douglas [5]. In this work, we have provided the complete classification of partial Noether operators for the general linear system of two second-order ODEs and all the first integrals are constructed with the help of partial Noether operators via a partial Lagrangian. The results for the special case $b(x) = c(x)$ of this system was considered elsewhere [21]. The linear system is self-adjoint in this special case and a Lagrangian exists. This was also reviewed in this paper. These equations model important physical phenomena in dynamics such as oscillator systems and are thus important. These form four, five and eight-dimensional subalgebras of the Lie algebras of the linear system of two second-order ODEs studied in [17]. This study provides a new way to construct first integrals for equations for which we do not have Lagrangians as partial Lagrangians do exist for such second-order equations in the absence of standard Lagrangians.

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